Quantum Theory of Many-Particle Systems, Phys. 540

- Class: M, W & F from 1:00 pm to 2:00 pm (Compton 241 mostly)
- HW: discussed after M&W class (if no conflict)
- Class goals:
  - learn how to do advanced non-relativistic quantum mechanics
  - describe and understand the main physics of interesting many-fermion and many-boson systems
- Prerequisite: at least one upper-level undergraduate quantum course (here 471)
- Grade: homework (40%), including computer assignments (20%), participation in classes (10%), and a presentation on a related subject of ~30 minutes (25%)
- No written exams
- Helpful to read some material in books on reserve in the library
Symmetric and antisymmetric states

When is quantum physics expected?

Consider the energy levels for a particle of mass $m$ enclosed in a box with volume $V = L^3$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2}{8mL^2} (n_x^2 + n_y^2 + n_z^2) \quad \text{positive integers}$$

Total number of states below energy $E$

$$\Omega(E) = \frac{\pi}{6} \left( \frac{8mL^2E}{h^2} \right)^{3/2} = \frac{\pi}{6} \left( \frac{8mE}{h^2} \right)^{3/2} V$$

"Quantumness" --> indistinguishability not important when

$$1 \gg Q \equiv \frac{N}{\Omega} = \frac{6}{\pi} \rho \left( \frac{h^2}{12mk_BT} \right)^{3/2}$$

Use $E = \frac{3}{2} k_BT$
<table>
<thead>
<tr>
<th>System</th>
<th>$T$ (K)</th>
<th>$\text{Density (m}^{-3}\text{)}$</th>
<th>Mass (u)</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>He (l)</td>
<td>4.2</td>
<td>$1.9 \times 10^{28}$</td>
<td>4.0</td>
<td>1.1</td>
</tr>
<tr>
<td>He (g)</td>
<td>4.2</td>
<td>$2.5 \times 10^{27}$</td>
<td>4.0</td>
<td>$1.4 \times 10^{-1}$</td>
</tr>
<tr>
<td>He (g)</td>
<td>273</td>
<td>$2.7 \times 10^{25}$</td>
<td>4.0</td>
<td>$2.9 \times 10^{-6}$</td>
</tr>
<tr>
<td>Ne (l)</td>
<td>27.1</td>
<td>$3.6 \times 10^{28}$</td>
<td>20.2</td>
<td>$1.1 \times 10^{-2}$</td>
</tr>
<tr>
<td>Ne (g)</td>
<td>273</td>
<td>$2.7 \times 10^{25}$</td>
<td>20.2</td>
<td>$2.5 \times 10^{-7}$</td>
</tr>
<tr>
<td>e(^{-}) Na metal</td>
<td>273</td>
<td>$2.5 \times 10^{28}$</td>
<td>$5.5 \times 10^{-4}$</td>
<td>$1.7 \times 10^{3}$</td>
</tr>
<tr>
<td>e(^{-}) Al metal</td>
<td>273</td>
<td>$1.8 \times 10^{29}$</td>
<td>$5.5 \times 10^{-4}$</td>
<td>$1.2 \times 10^{4}$</td>
</tr>
<tr>
<td>e(^{-}) white dwarfs</td>
<td>$10^7$</td>
<td>$10^{36}$</td>
<td>$5.5 \times 10^{-4}$</td>
<td>$8.5 \times 10^{3}$</td>
</tr>
<tr>
<td>p,n nuclear matter</td>
<td>$10^{10}$</td>
<td>$1.7 \times 10^{44}$</td>
<td>1.0</td>
<td>$6.5 \times 10^{2}$</td>
</tr>
<tr>
<td>n neutron star</td>
<td>$10^{8}$</td>
<td>$4.0 \times 10^{44}$</td>
<td>1.0</td>
<td>$1.5 \times 10^{6}$</td>
</tr>
<tr>
<td>$^{87}\text{Rb condensate}$</td>
<td>$10^{-7}$</td>
<td>$10^{19}$</td>
<td>87</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Bosons and Fermions

- Use experimental observations to conclude about consequences of identical particles
- Two possibilities
  - antisymmetric states $\Rightarrow$ fermions half-integer spin
    - Pauli from properties of electrons in atoms
  - symmetric states $\Rightarrow$ bosons integer spin
    - Considerations related to electromagnetic radiation (photons)
- Can also consider quantization of “field” equations
  - e.g. quantize “free” Maxwell equations
Wolfgang Pauli (1900-1958)

- The Nobel Prize in Physics 1945 was awarded to Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle".

- paper Zeitschr. f. Phys. 31, 765 (1925)
Review single-particle states

- **Notation**  
  $$|...\rangle$$

- ... list of quantum numbers associated with a CSCO

- **Normalization**  
  $$\langle \alpha | \beta \rangle = \delta_{\alpha,\beta}$$

- **Continuous quantum numbers**
  - **Example**  
    $$\langle \mathbf{r}, m_s | \mathbf{r}', m'_s \rangle = \delta(\mathbf{r} - \mathbf{r}') \delta_{m_s, m'_s}$$

- **Completeness**  
  $$\sum_{\alpha} |\alpha\rangle \langle \alpha| = 1$$

**Consequences for two-particle states**

- **CVS for two particles: product space**

- **Notation**  
  $$|\alpha_1 \alpha_2 \rangle = |\alpha_1 \rangle |\alpha_2 \rangle$$

- **Orthogonality**  
  $$(\alpha_1 \alpha_2 | \alpha'_1 \alpha'_2 \rangle = \delta_{\alpha_1, \alpha'_1} \delta_{\alpha_2, \alpha'_2}$$

- **Completeness**  
  $$\sum_{\alpha_1 \alpha_2} |\alpha_1 \alpha_2 \rangle (\alpha_1 \alpha_2 | = 1$$
Exchange degeneracy

- Consider \( \alpha_1 \neq \alpha_2 \)
- Then \( |\alpha_2\alpha_1\rangle \neq |\alpha_1\alpha_2\rangle \)
- All states \( |\alpha_1\alpha_2\rangle \)
  \( |\alpha_2\alpha_1\rangle \)
  \( c_1 |\alpha_1\alpha_2\rangle + c_2 |\alpha_2\alpha_1\rangle \)

yield \( \alpha_1 \) for one particle and \( \alpha_2 \) for the other upon measurement.
- Yet, unclear which state describes this system and therefore inconsistent with quantum postulates.
- Consider permutation operator
  \[ P_{12} |\alpha_1\alpha_2\rangle = |\alpha_2\alpha_1\rangle \]

with \( P_{12} = P_{21} \) and \( P_{12}^2 = 1 \)
- Hamiltonian for two particles is symmetric for \( 1 \Leftrightarrow 2 \)
Development

• Typical Hamiltonian

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(|r_1 - r_2|) \]

• Consider operator acting on particle 1 and corresponding eigenvalue

\[ A_1|\alpha_1 \alpha_2) = a_1|\alpha_1 \alpha_2) \]

• Similarly, the same operator acting on particle 2 yields

\[ A_2|\alpha_1 \alpha_2) = a_2|\alpha_1 \alpha_2) \]

• Note

\[ P_{12} A_1|\alpha_1 \alpha_2) = a_1 P_{12}|\alpha_1 \alpha_2) = a_1|\alpha_2 \alpha_1) = A_2|\alpha_2 \alpha_1) \]

• and

\[ P_{12} A_1|\alpha_1 \alpha_2) = P_{12} A_1 P_{12}^{-1} P_{12}|\alpha_1 \alpha_2) = P_{12} A_1 P_{12}^{-1}|\alpha_2 \alpha_1) \]

• Holds for any state; therefore

\[ P_{12} A_1 P_{12}^{-1} = A_2 \]

• It follows that

\[ P_{12} H P_{12}^{-1} = H \quad \text{or} \quad [P_{12}, H] = 0 \]
Symmetric and antisymmetric two-particle states

- So \([P_{12}, H] = 0\)

- Common eigenkets either

  \[|\alpha_1\alpha_2\rangle_+ = \frac{1}{\sqrt{2}} \{ |\alpha_1\alpha_2\rangle + |\alpha_2\alpha_1\rangle \} \]

  or

  \[|\alpha_1\alpha_2\rangle_- = \frac{1}{\sqrt{2}} \{ |\alpha_1\alpha_2\rangle - |\alpha_2\alpha_1\rangle \} \]

- Eigenstates of the Hamiltonian either symmetric \(\Rightarrow\) bosons
  or antisymmetric \(\Rightarrow\) fermions

- Two-boson state

  \[|\alpha_1\alpha_2\rangle_S = \left[ \frac{1}{2n_\alpha!n_\alpha'!...} \right]^{1/2} \{ |\alpha_1\alpha_2\rangle + |\alpha_2\alpha_1\rangle \} \]

  \[\alpha_1 = \alpha_2 = \alpha \Rightarrow |n_\alpha = 2\rangle = |\alpha\alpha\rangle_S = |\alpha\rangle |\alpha\rangle \]

  \[\alpha_1 \neq \alpha_2 \Rightarrow |\alpha_1\alpha_2\rangle_S = \frac{1}{\sqrt{2}} \{ |\alpha_1\alpha_2\rangle + |\alpha_2\alpha_1\rangle \} \]
Fermions

- **Antisymmetry:** \(|\alpha_2 \alpha_1\rangle = -|\alpha_1 \alpha_2\rangle\)

- Both kets represent the same physical state: count only once in completeness relation \(\Rightarrow\) “order” quantum numbers
  \(|1\rangle, |2\rangle, |3\rangle, ...\)

- **Ordered:** \(\sum_{i<j} |ij\rangle \langle ij| = 1\)

- **Not ordered:** \(\frac{1}{2!} \sum_{ij} |ij\rangle \langle ij| = 1\)

Bosons ordered: \(\sum_{i<j} |ij\rangle \langle ij| = 1\)

not ordered: \(\sum_{ij} \frac{n_1!n_2!...}{2!} |ij\rangle \langle ij| = 1\)
Scattering of identical particles

Particles that can be “distinguished”

particle a in D1 (a) \[ \frac{d\sigma}{d\Omega} (a \text{ in } D_1, b \text{ in } D_2) = |f(\theta)|^2 \]

particle a in D2 (b) \[ \frac{d\sigma}{d\Omega} (a \text{ in } D_2, b \text{ in } D_1) = |f(\pi - \theta)|^2 \]

any particle in D1 \[ \frac{d\sigma}{d\Omega} (\text{particle in } D_1) = |f(\theta)|^2 + |f(\pi - \theta)|^2 \]
Identical bosons

- Cannot distinguish (a) and (b)

- Rule for bosons: add amplitudes then square!

  $\frac{d\sigma}{d\Omega} (\text{bosons}) = |f(\theta) + f(\pi - \theta)|^2$

- Interference

- 90 degrees: factor of 2 compared to “classical” cross section
$^{12}\text{C} + ^{12}\text{C}$

Low-energy boson-boson scattering

$^{12}\text{C}$ a boson?

- 6 protons and 6 neutrons
- total angular momentum integer (made of 12 spin-$\frac{1}{2}$ particles)
- ground state $0^+$
- first excited state above 4 MeV

- $^4\text{He}$ atom: $2p + 2n + 2e \Rightarrow$ boson
- $^3\text{He}$ atom: $2p + 1n + 2e \Rightarrow$ fermion
- but
Fermion-fermion scattering

- Identical fermions: electrons with spin up

\[
\frac{d\sigma}{d\Omega}(\text{fermions}) = |f(\theta) - f(\pi - \theta)|^2
\]

- What about