1. (a) Setting the determinant of the matrix
\[
\begin{bmatrix}
  b - \lambda & 0 & 0 \\
  0 & -\lambda & -ib \\
  0 & ib & -\lambda \\
\end{bmatrix}
\]
equal to zero and finding all of the values of \( \lambda \) gives you the eigenvalues of the matrix. Doing this you find that you get the value of \( \lambda = b \) twice, meaning that this operator exhibits a degenerate spectrum.

(b) Working out the commutation relation \( ([A, B] = AB - BA) \) through matrix multiplication you are able to show that \([A, B] = 0 \) meaning that \( A \) and \( B \) commute.

(c) First of all you must find basis kets of \( B \). This is done by letting the matrix \( B \) operate on a general combination of kets, described in the given basis, and returning an eigenvalue times the original ket.

\[
\begin{bmatrix}
  b & 0 & 0 \\
  0 & 0 & -ib \\
  0 & ib & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma \\
\end{bmatrix} = \lambda \begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma \\
\end{bmatrix}.
\]
This operation gives you a set of equations which when used with the normalization condition can be solved for the set of unknowns \((\alpha, \beta, \gamma)\). Once you have solved for the unknowns you have the coefficients which multiply the kets in the given basis to form the kets in the new basis. The eigenkets of \( B \) which you find turn out to be

\[
|\alpha_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
|\alpha_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix},
|\alpha_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}.
\]
If you let \( A \) operate on these new kets you see that these kets are in fact eigenkets of \( A \). Each one of these kets has a unique set of eigenvalues of \( A \) and \( B \), thus the eigenkets are completely characterized.

2. (a) In the \( S_z \) basis \( S_z |+\rangle = \frac{n}{2} |+\rangle \) and \( S_z |\pm\rangle = \frac{n}{2} |\pm\rangle \) thus \( \langle S_z \rangle = 0 \) and \( \langle S_z^2 \rangle = \frac{n^2}{4} \).
yielding $\langle (\Delta S_x)^2 \rangle = \hbar^2 \frac{\alpha^2}{4}$. In a similar fashion you see that $\langle (\Delta S_y)^2 \rangle = \hbar^2 \frac{\alpha^2}{4}$. From this you arrive at $\langle (\Delta S_x)^2 (\Delta S_y)^2 \rangle = \hbar^4 \frac{1}{16}$. The commutator of $S_x$ and $S_y$ yields a value of $\frac{i\hbar^2}{2}$ meaning that $\frac{1}{4}|\langle [S_x, S_y] \rangle|^2 = \hbar^4 \frac{1}{16}$ thus the generalized uncertainty relation is satisfied.

(b) Here the spread in $S_x$ is zero, but so is the absolute square of commutator of $S_x$ and $S_y$ thus you arrive a $0 \geq 0$ which is true, so again you see that the uncertainty relation is satisfied.

3. Taking expectation values as usual you arrive at $\langle S_x \rangle^2 = \hbar^2 \alpha^2(1 - \alpha^2) \cos^2 \beta$, $\langle S_y \rangle^2 = \hbar^2 \alpha^2(1 - \alpha^2) \sin^2 \beta$, $\langle S_x^2 \rangle = \frac{\hbar^2}{4} = \langle S_y^2 \rangle$. These values lead to $\langle (\Delta S_x)^2 (\Delta S_y)^2 \rangle = \hbar^4 \left( \frac{1}{16} - \frac{\alpha^2}{4}(1 - \alpha^2) + \frac{1}{4}\alpha^4(1 - \alpha^2)^2 \sin^2(2\beta) \right)$. You can also see that this function is maximized when $\langle S_x \rangle^2 = \langle S_y \rangle^2 = 0$. This occurs when $\alpha = -1, 0, 1$. Each of these values of $\alpha$ has a corresponding ket which will maximize our product. The three kets which maximize our product are $|+\rangle, -|+\rangle, e^{i\beta}|-\rangle$. For each one of these kets you get $\langle (\Delta S_x)^2 (\Delta S_y)^2 \rangle = \hbar^4 \frac{1}{16}$. You also see that $\frac{1}{4}|\langle [S_x, S_y] \rangle|^2 = \hbar^4 \frac{1}{16}$ thus the uncertainty principle holds.

As a side note many people got $\alpha = 1/\sqrt{2}$ to be the value which maximizes this function. They did this by setting the derivative of the function equal to zero and solving for $\alpha$. If this process is taken one step farther (i.e. the second derivative test) you see that the second derivative of this function evaluated at $\alpha = 1/\sqrt{2}$ is a positive number meaning that a local minimum occurs in this function at that point, not a maximum.