Anharmonic oscillators and conformal field theories

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St. Louis, March 2009
Outline

1. 2D quantum conformal field theory and the minimal models
2. The Kac table
3. The ODE/IM correspondence for the minimal models
3. Monodromy properties and the Kac table
3. Conclusions
2D quantum conformal field theories play a central role in:
- critical phenomena in 2D condensed matter physics,
- String theory.

The ferromagnetic Ising model Hamiltonian is

$$H[\sigma] = - \sum_{<ij>} \sigma_i \sigma_j, \quad \sigma_i = \pm 1$$

the sum is over nearest neighbor sites. The model exhibits:
- for $T > T_c$: a disordered phase with $<\sigma> = 0$,
- for $T < T_c$: an ordered phase with $<\sigma> \neq 0$,
- at $T = T_c$ a second order phase transition. At $T = T_c$ there are fluctuations at all length scales and the continuum limit version of the model is conformal invariant.
2D conformal transformations coincide with the analytic transformation

\[ z \rightarrow f(z) \ , \ \bar{z} \rightarrow \bar{f}(\bar{z}) \]

with \( z = x + iy \), \( \bar{z} = x - iy \).

The corresponding infinitesimal generators are

\[ l_n = -z^{n+1} \partial_z \ , \ \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}} \ , \]

They satisfy

\[ [l_m, l_n] = (m - n)l_{m+n} \ , \ [\bar{l}_m, \bar{l}_n] = (m - n)\bar{l}_{m+n} \ , \ [l_n, \bar{l}_m] = 0 \ . \]

At quantum level \( l_m \rightarrow L_m \), \( \bar{l}_m \rightarrow \bar{L}_m \), \( [L_n, \bar{L}_m] = 0 \) and

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} (n^3 - n)\delta_{n+m,0} \]

\( c \) : central charge/conformal anomaly/Casimir coefficient.
Primary operators: special operators that transform in a simple way: \( \phi(z, \bar{z}) \rightarrow (\partial_z f)^h (\partial_{\bar{z}} \bar{f})^{\bar{h}} \phi(f, \bar{f}) \).

All the operators can be obtained from the primary operators by acting with \( \{ L_{-n}, \bar{L}_{-n} \} \).

For generic \( c \) there is an \( \infty \) number of primary operators.

Given two co-prime integers \( p < q \) the Minimal Models \( \mathcal{M}_{p,q} \) are CFTs with

\[
c = 1 - 6 \frac{(p - q)^2}{pq} < 1
\]

and only a finite number of primary operators. They have conformal dimensions

\[
h_{s,r} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \quad (1 \leq s < p, 1 \leq r < qs/p)
\]
The set of $h_{r,s}$ form the so-called Kac table.

- $\mathcal{M}_{3,4} \equiv$ Ising model:
  
  $$c = 1/2 \ , \ \mathbb{1} \leftrightarrow h_{1,1} = 0 \ , \ \epsilon \leftrightarrow h_{1,3} = 1/2 \ , \ \sigma \leftrightarrow h_{1,2} = 1/16$$

- $\mathcal{M}_{2,5} \equiv$ Yang-Lee model:

  $$c = -22/5 \ , \ \mathbb{1} \leftrightarrow h_{1,1} = 0 \ , \ \phi \leftrightarrow h_{1,2} = -1/5 .$$
The ODE/IM correspondence for the minimal models

The Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + (x^2M - E) + \frac{l(l+1)}{x^2}\right) \psi(x) = 0,$$

with \(\psi(0) = \psi(\infty) = 0\), \(M > 0\) and \(l\) real, is related to CFT.

- The associated Stokes relations imply constraints on its eigenvalues \(E \in \{E_i\}\) which coincide with the Bethe Ansatz Equations for the 6-vertex model in its conformal (\(c = 1\)) limit.
- The same BAEs emerge from the study of \(c \leq 1\) CFTs in the framework developed by Bazhanov, Lukyanov and Zamolodchikov.

The Schrödinger equation is related to a CFT and a primary operator with

\[c = 1 - \frac{6M^2}{M+1}, \quad h = \frac{(2l + 1)^2 - 4M^2}{16(M+1)}\].
For any two coprime integers $p < q$, the ground state of the minimal model $\mathcal{M}_{p,q}$ is found by setting

$$M + 1 = \frac{q}{p}, \quad l + \frac{1}{2} = \frac{1}{p}$$

in the Schrödinger equation. This corresponds to the central charge

$$c_{pq} = 1 - \frac{6}{pq}(q-p)^2$$

and lowest-possible conformal dimension

$$h = \frac{4}{pq}(1 - (q-p)^2)$$

- **Ising**: $\mathcal{M}_{3,4}$, $h = 0 \leftrightarrow 1$
- **Yang-Lee**: $\mathcal{M}_{2,5}$, $h = -1/5 \leftrightarrow \phi$
The \((l(l+1))/x^2\) term can be eliminated by the simple transformation

\[ x = z^{p/2}, \quad \psi(x, E) = z^{p/4-1/2}y(z, E), \]

and the rescaling \( z \rightarrow (2/p)^{2/q}z : \)

\[ \left( -\frac{d^2}{dz^2} + z^{p-2}(z^{q-p} - \tilde{E}) \right) y(z, \tilde{E}) = 0 \]

where

\[ \tilde{E} = \left( \frac{p}{2} \right)^{2-2p/q}E. \]

- The change of variable has replaced a singular potential defined on a multi-sheeted Riemann surface, by a simple polynomial.
- Any solution to the transformed equation is automatically single-valued around \( z = 0 \).
To see which other primary states have similarly-trivial monodromy, perform the same transformation with $l > -1/2$:

$$\left(-\frac{d^2}{dz^2} + \frac{\tilde{l}(\tilde{l}+1)}{z^2} + z^{p-2}(z^{q-p} - \tilde{E})\right)y(z, \tilde{E}, \tilde{l}) = 0$$

where

$$2(\tilde{l} + \frac{1}{2}) = p(l + \frac{1}{2}).$$

The Fuchsian singularity at $z = 0$ means that the equation admits a pair of solutions

$$\chi_1(z) = z^{\lambda_1} \sum_{n=0}^{\infty} c_n z^n ; \quad \chi_2(z) = z^{\lambda_2} \sum_{n=0}^{\infty} d_n z^n ,$$

where $\lambda_1 = \tilde{l} + 1 > \lambda_2 = -\tilde{l}$ are the two roots of the indicial equation $\lambda(\lambda - 1) - \tilde{l}(\tilde{l} + 1) = 0$ and

$$\chi_j(e^{2\pi i} z) = e^{2\pi i \lambda_j} \chi_j(z) , \quad j = 1, 2.$$
A general solution is

\[ y(z, \tilde{E}, \tilde{l}) = \sigma \chi_1(z) + \tau \chi_2(z) \]

and we shall demand that the monodromy of \( y(z) \) around \( z = 0 \) is projectively trivial:

\[ y(e^{2\pi i} z) \propto y(z). \]

This condition imposes

i) \( 2\tilde{l} + 1 \) is a positive integer;

ii) The allowed values of \( 2\tilde{l} + 1 \) form the set of holes of the infinite sequence

\[ pr + qs \quad , \quad r, s = 0, 1, 2, 3 \ldots \quad (*) . \]

We shall call the set of integers \( (*) \) ‘representable’ and denote them by \( \mathbb{R}_{pq} \).

As a consequence

\[ h = \frac{(2\tilde{l} + 1)^2 - (p - q)^2}{4pq} \]

reproduces the set of conformal weights of the primary operators in the Kac table of \( \mathcal{M}_{p,q} \).
Ising model \( \{3r + 4s\} \): 

\[
\begin{array}{ccccccc}
0 & \frac{1}{16} & & & & & \\
\bullet & \circ & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

(The holes in the infinite sequence of integers for the critical Ising model \( \mathcal{M}_{3,4} \).
The holes are at 1, 2 and 5.)

Yang-Lee model \( \{2r + 5s\} \): 

\[
\begin{array}{ccccccc}
\frac{-1}{5} & 0 & & & & & \\
\bullet & \circ & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

(Holes for the Lee-Yang model \( \mathcal{M}_{2,5} \), at 1 and 3.)
To establish these claims:

- The general solution $y(z)$ be projectively trivial means that $\chi_1(z)$ and $\chi_2(z)$ must have the same monodromy, which implies that:

  $$\lambda_1 - \lambda_2 = 2\tilde{l} + 1 \in \mathbb{N}.$$ 

- This in turn restricts $\tilde{l}$ to be an integer or half integer, so that, naively, the allowed solutions are even or odd under a $2\pi i$ rotation around $z = 0$.

- In such a circumstance, while $\chi_1(z)$ keeps its power series expansion, $\chi_2(z)$ generally acquires a log contribution:

  $$\chi_2(z) = D\chi_1(z) \log(z) + \frac{1}{z\tilde{l}} \sum_{n=0}^{\infty} d_n z^n.$$ 

- Unless $D = 0$, this will spoil the projectively trivial monodromy of $y(z)$. 

The log term is absent iff the recursion relation for the $d_n$'s with $D = 0$

$$n(n - 2\tilde{l} - 1) d_n = d_{n-q} - \tilde{E} d_{n-p}$$

with the initial conditions $d_0 = 1$, $d_{m<0} = 0$ admits a solution. Consider first the case $2\tilde{l} + 1 \notin \mathbb{R}_{pq}$. Starting from the given initial conditions, the recursion relation generates a solution of the form

$$\chi_2(z) = \frac{1}{z^{\tilde{l}}} \sum_{n=0}^{\infty} d_n z^n$$

where the only nonzero $d_n$'s are those for which the label $n$ lies in the set $\mathbb{R}_{pq}$. Given that $2\tilde{l} + 1 \notin \mathbb{R}_{pq}$, for these values of $n$ the factor $n(n - 2\tilde{l} - 1)$ on the LHS of the recursion relation is never zero, and the procedure is well-defined.
If instead $2\tilde{l} + 1 \in \mathbb{R}_{pq}$, then the recursion equation taken at $n = 2\tilde{l} + 1$ yields the additional condition

$$\tilde{E} \, d_{2\tilde{l}+1-p} - d_{2\tilde{l}+1-q} = 0,$$

which is inconsistent for generic $\tilde{E}$, and so the log term is required.

Given the characterisation of $\mathbb{R}_{pq}$, the set $\mathbb{Z}^+$ of non-negative integers can be written as a disjoint union

$$\mathbb{Z}^+ = \mathbb{R}_{pq} \cup \mathbb{N}_{pq}$$

where $\mathbb{N}_{pq}$ is the set of ‘nonrepresentable’ integers. If the coprime integers $p$ and $q$ are larger than 1 then $\mathbb{N}_{pq}$ is non-empty; in fact

$$|\mathbb{N}_{pq}| = \frac{1}{2}(p-1)(q-1),$$

a result which goes back to Sylvester.
Conclusions

- Carl is the best!!