RESEARCH PROGRAM OF CARL M. BENDER

Prof. Bender's scholarly expertise is in mathematical physics and applied mathematics. Since the publication with S. Orszag of his influential book, *Advanced Mathematical Methods for Scientists and Engineers*, he has been recognized as an expert on the subject of asymptotic analysis, differential equations, and perturbative methods and their use in solving problems in theoretical physics.

Bender has made many pioneering discoveries in mathematical and theoretical physics in his 40-year career by applying complex-variable and differential-equation theory to gain a deeper understanding of the mathematical structure of quantum mechanics and quantum field theory. Below is a detailed analysis of his research accomplishments in chronological order beginning with his early work on the anharmonic oscillator and continuing through his dramatic discoveries of the past decade of $\mathcal{PT}$ symmetric quantum mechanics and quantum field theory. The order of this presentation is intended to give an annotated explanation of the full scope and depth of Prof. Bender's research. There is particular emphasis on Prof. Bender's recent and ongoing discoveries on $\mathcal{PT}$ symmetry. A detailed description of work in progress and future projects and objectives is given.

1. The Anharmonic Oscillator

Bender's first major work used the quantum anharmonic oscillator to elucidate the nature of perturbation theory. This work, in collaboration with T. T. Wu, explained for the first time why perturbation series diverge. The divergence is caused by singularities, known as *Bender-Wu singularities*, in the complex-coupling-constant plane. His first papers on this subject are

"Analytic Structure of Energy Levels in a Field-Theory Model"
C. M. Bender and T. T. Wu, Physical Review Letters **21**, 406 (1968)
"Anharmonic Oscillator"
C. M. Bender and T. T. Wu, Physical Review **184**, 1231 (1969)

The PRL has 102 citations and the PR paper has 656 citations listed in the Web of Science data base. As a result of Prof. Bender's work, the anharmonic oscillator is now a standard testing ground for new ideas and techniques in quantum field theory and quantum mechanics.

2. Large-Order Behavior of Perturbation Theory

Prof. Bender's next major contribution, also in collaboration with T. T. Wu, originated the study of the large-order behavior of perturbation theory. Bender and Wu showed that this asymptotic limit reflects the underlying semiclassical properties of a quantum theory. They obtained their results using WKB analysis, statistical analysis of Feynman graphs, asymptotic analysis of difference equations, instanton methods, and dispersion relations. Their semiclassical dispersion-relation approach to finding the asymptotic behavior of perturbation coefficients is known as *Bender-Wu theory*. The principal ideas of Bender-Wu theory were introduced in

"Large-Order Behavior of Perturbation Theory"
C. M. Bender and T. T. Wu, Physical Review Letters **27**, 461 (1971)
"Anharmonic Oscillator. II. A Study of Perturbation Theory in Large Order"
C. M. Bender and T. T. Wu, Physical Review D **7**, 1620 (1973)
Reprinted in: J. C. Le Guillou and J. Zinn-Justin (eds.),
Large-order behaviour of perturbation theory, (North-Holland, 1990), p. 41-57
"Statistical Analysis of Feynman Diagrams"
C. M. Bender and T. T. Wu, Physical Review Letters **37**, 117 (1976)

As a result of Prof. Bender's pioneering work, the field of large-order behavior of perturbation theory has become an active research area, with contributions from numerous groups in quantum field theory, atomic physics, chemical physics, and mathematical physics. Prof. Bender was the keynote speaker at a Sanibel Symposium on Perturbation Theory in Large Order, a conference devoted entirely to this field that he created. The Sanibel Symposium was attended by researchers from 25 nations. (Main speakers included T. T. Wu, B. Simon, E. Brezin, S. Graffi, V. Grecchi, N. Khuri, W. Reinhardt, J. Zinn-Justin, and B. Nickel.)

The fundamental work of Bender and Wu on the large-order behavior of perturbation theory led to the development of instanton techniques. It has also evolved into a new and active area of research called hyperasymptotics, or asymptotics beyond all orders. A six-month workshop (winter and spring 1995) on this area was held at the Isaac Newton Institute in Cambridge. This workshop was organized by M. Berry (Bristol) and M. Kruskal (Princeton), and Prof. Bender was one of the principal invited participants.

3. Multidimensional Semiclassical Methods

Banks, Bender, and Wu developed powerful multidimensional semiclassical techniques in their work on the large-order behavior of perturbation theory. These methods are now widely used. S. Coleman, for example, used these methods in his work on the decay of the false vacuum. These semiclassical methods were introduced in

"Coupled Anharmonic Oscillator. I. Equal Mass Case"
C. M. Bender, T. I. Banks, and T. T. Wu, Physical Review D **8**, 3346 (1973)
"Coupled Anharmonic Oscillators. II. Unequal Mass Case"
C. M. Bender and T. I. Banks, Physical Review D **8**, 3366 (1973)

4. Development of New Perturbative and Nonperturbative Techniques

Bender introduced and studied in depth many perturbative and nonperturbative methods that are now routinely used in quantum mechanics and quantum field theory. These methods include strong-coupling techniques, high-temperature lattice approximations, finite-element approximations, dimensional expansions, and multiple-scale analysis:

"Effective Potential for a Renormalized d-Dimensional $g\phi^4$ Field Theory in the $g \to \infty$ Limit"
C. M. Bender, F. Cooper, G. S. Guralnik, H. Moreno, R. Roskies, and D. H. Sharp
Physical Review Letters **45**, 501 (1980)
"Solution of Operator Field Equations by the Method of Finite Elements"
C. M. Bender and D. H. Sharp, Physical Review Letters **50**, 1535 (1983)
"Resolution of the Operator-Ordering Problem Using the Method of Finite Elements"
C. M. Bender, L. R. Mead, and S. S. Pinsky
Physical Review Letters **56**, 2445 (1986)
"Dimensional Expansions"
C. M. Bender, S. Boettcher, and L. Lipatov, Physical Review Letters **68**, 3674 (1992)
"Multiple-Scale Analysis of the Quantum Anharmonic Oscillator"
C. M. Bender and L. M. A. Bettencourt, Physical Review Letters **77**, 4114 (1996)

A note on dimensional expansions: This idea, developed in collaboration with L. Lipatov, is to expand the Green's functions of the quantum field theory as series in powers of the space-time dimension D. The leading term in such an expansion is easy to calculate because it requires that one solve the corresponding zero-dimensional field theory. However, it is not obvious how to calculate higher terms in such a series. In the Physical Review Letter above, Bender calculated the coefficient of D in this dimensional expansion in *closed form* for a ϕ^{2N} quantum field theory. Moreover, he developed graphical rules (resembling those used in high-temperature lattice calculations) that give *all* higher terms in the series. The D expansion has a finite radius of convergence, and studies show that only a small number of terms are needed to obtain accurate results.

Bender has applied dimensional expansion methods to a variety of physical problems. For example, with S. Boettcher (his former graduate student) he studied the Ising limit of quantum field theory and calculated the D expansion of the *renormalized* Green's functions in the Ising limit. In another investigation, Bender and K. Milton elucidated the dimensional dependence of the Casimir force by calculating the stress on a D-dimensional spherical shell due to a scalar quantum field. Prof. Bender has also undertaken dimensional studies of classical processes. Prof. Bender published seven papers, including a PRL on random walks in D-dimensional space:

"Universality in Random Walks with Birth and Death"
C. M. Bender, S. Boettcher, and P. N. Meisinger
Physical Review Letters **75**, 3210 (1995)

In these papers Prof. Bender showed how to define a random walk in D dimensions (where D is not necessarily an integer!). He showed that random walks which allow for the creation and annihilation of random walkers exhibit critical behavior, and he calculated the critical index in closed form as a function of the dimension D. This work has experimental implications for polymer growth; it predicts the critical index that describes the phase transition, called an *adsorption transition*, that occurs when a polymer is growing in the region of an adsorbing surface.

5. The Delta Expansion

The most unusual approximation method developed by Prof. Bender, called the *delta expansion*, is applicable in both classical and quantum physics. The delta expansion is a powerful tool for solving nonlinear problems and it yields a *convergent* perturbation series. It was introduced in

"Logarithmic Approximations to Polynomial Lagrangians"
C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr.
Physical Review Letters **58**, 2615 (1987)

The theme of the delta expansion is to expand in the *power* of the interaction term. Thus, to solve a $\lambda\phi^4$ quantum field theory in D-dimensional space-time, one introduces a small parameter δ and considers a $\lambda(\phi^2)^{1+\delta}$ field theory. The parameter δ is thus a measure of the nonlinearity of the interaction term. One then calculates the Green's functions as series in powers of δ. The advantages of this method are that (1) all n-point Green's functions have formal power series in powers of δ; (2) there is a diagrammatic procedure for computing each coefficient in the δ series; (3) the δ series has a finite radius of convergence, which for the above example is 1; (4) numerical results in low-dimensional models, for which the answers are known, are superb.

Bender and H. Jones used the delta expansion to show that when the dimension of space-time is greater than or equal to four, a ϕ^4 field theory becomes trivial ["Renormalization and the Triviality of $(\lambda\phi^4)_4$ Field Theory," C. M. Bender and H. F. Jones, Phys. Rev. D **38**, 2526 (1988)]. This is a major result in quantum field theory.

The δ-series provides a natural way to study supersymmetric quantum field theories because inserting the parameter δ does not break global supersymmetry invariance. Also, the δ expansion gives beautiful results for high-temperature field theory. Because of infrared divergences, ordinary expansions in high-temperature field theory often fail at a small finite order in perturbation theory. The δ expansion has no infrared divergences and such problems are completely avoided. The numerical results in such delta expansions are superb. The delta expansion is also useful in the study of stochastic quantization and Prof. Bender obtained excellent results for the Langevin equation.

The δ-series is not only for quantum physics; it is a general perturbative technique that solves many kinds of nonlinear problems. Prof. Bender showed that the δ-series can be used to solve well-known intractable nonlinear problems in many branches of mathematical physics. Remarkably accurate results were easily obtained for the Thomas-Fermi, Blasius, Lane-Emden, and Duffing equations, and nonlinear problems in electron transport theory. He applied the same methods to nonlinear partial differential equations such as the Burgers and Korteweg-de Vries equations.

6. $\mathcal{PT}$-Symmetric Quantum Theory

The development of the delta expansion led directly to Prof. Bender's most recent work on $\mathcal{PT}$-symmetric quantum mechanics:

"Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry"
C. M. Bender and S. Boettcher
Physical Review Letters **80**, 5243-5246 (1998)

"Complex Extension of Quantum Mechanics"
C. M. Bender, D. C. Brody, and H. F. Jones
Physical Review Letters **89**, 270401 (2002)

"Scalar Quantum Field Theory with Complex Cubic Interaction"
C. M. Bender, D. C. Brody, and H. F. Jones
Physical Review Letters **93**, 251601 (2004)

"Faster than Hermitian Quantum Mechanics"
C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister
Physical Review Letters **98**, 040403 (2007)

The first of these letters has 387 citations on the Web of Science data base and has generated a huge industry. New papers on $\mathcal{PT}$ symmetry appear on the web arXiv almost every day. There have been many theses written and over half a dozen international conferences have been held on this subject. The last international conference, which was held in London in July 2007, had nearly 100 registered participants.

Bender originated the study of complex quantum mechanics and the theory of non-Hermitian Hamiltonians in 1998. This is a new area of theoretical physics in which the quantum theories that are obtained may be regarded as extensions of conventional quantum mechanics and quantum field theory into the complex domain. Complex quantum mechanics is not just a mathematical breakthrough; it is a major advance in the basic theory of quantum mechanics. The potential for research in this area is immense. Complex quantum mechanics provides a framework for describing the nature of antiparticles and offers the possibility that a particle and its corresponding antiparticle need not have identical masses, and thus it may provide insight into the puzzle of the baryon-antibaryon asymmetry of the universe. Complex quantum mechanics gives a setting for exploring the physics of the Higgs particle. Furthermore, complex quantum mechanics offers a possible mechanism to explain the phenomenon of dark energy.

Bender's discovery of complex quantum mechanics was announced in a 1998 Physical Review Letter, which reported an astonishing new result: A quantum-mechanical Hamiltonian need not be Hermitian to have a spectrum that is entirely real and positive. (The term *Hermitian* is used in the Dirac sense, where Hermitian conjugate means combined transpose and complex conjugate.) Some amazing examples of non-Hermitian quantum-mechanical Hamiltonians having entirely real, positive, and discrete spectra are $H = p^2 + ix^3$ and $H = p^2 - x^4$. These are special cases of a general class of Hamiltonians of the form $H = p^2 + x^2(ix)^\epsilon$, where ϵ is a real parameter.

Bender's 1998 Physical Review Letter argued that the reality and the positivity of the spectrum of H for $\epsilon > 0$ are a consequence of its *space-time reflection symmetry* ($\mathcal{PT}$ symmetry). When $\epsilon < 0$, the $\mathcal{PT}$ symmetry is spontaneously broken and the spectrum is not real. Conventional Hermitian quantum mechanics sits at the transition point between the regions of unbroken and broken $\mathcal{PT}$ symmetry. The Letter proposed that the condition of Hermiticity can be replaced by the more physical requirement of $\mathcal{PT}$ symmetry. It argued that Hermiticity is a convenient mathematical condition, but one whose physical basis is obscure and remote. Thus, the letter conjectured that the full symmetry group of the universe is the (continuous) proper Lorentz group and the (discrete) space-time reflection symmetry group. As $\mathcal{PT}$ symmetry is a different requirement from conventional Dirac Hermiticity, Hamiltonians that previously would have been rejected can now be considered as potentially valid descriptions of physical processes. Also, because a $\mathcal{PT}$-symmetric Hamiltonian may be complex, this proposal extends and generalizes the ideas of classical mechanics and dynamical systems, integrability, and chaos into the complex domain as well. These ideas are described in detail in two invited review papers:

"Introduction to $\mathcal{PT}$-Symmetric Quantum Theory"

C. M. Bender, Contemp. Phys. **46**, 277 (2005) (arXiv: quant-ph/0501052)

"Making Sense of Non-Hermitian Hamiltonians"

C. M. Bender, Rep. Prog. Phys. **70** (2007) 947-1018 (arXiv: hep-th/0703096)

Bender has now published well over fifty refereed papers on $\mathcal{PT}$ symmetry and complex non-Hermitian Hamiltonians. These papers investigate the consequences and ramifications of complex quantum mechanics and quantum field theory. Prof. Bender has worked in collaboration with many theoretical physicists including M. Berry, D. Brody, F. Cooper, G. Dunne, J. Feinberg, H. Jones, K. Milton, R. Rivers, E. Weniger, and more than a dozen graduate students. These new ideas have been extended to new kinds of quantum-mechanical and quantum-field-theoretic models by scores of mathematical and theoretical physicists who have been inspired by the ideas of $\mathcal{PT}$ symmetry.

In the first few years after the discovery of $\mathcal{PT}$-symmetric Hamiltonians it was observed that $\mathcal{PT}$-symmetric quantum theories have many remarkable and unexpected properties. For example, while a conventional Hermitian $g\phi^4$ quantum field theory has a repulsive potential and thus has no bound states, a $\mathcal{PT}$-symmetric $-g\phi^4$ theory has an attractive potential and does indeed have bound states. What is surprising is that, as Prof. Bender showed in a Physics Letter, the bound states become more *weakly* bound as the coupling strength g increases. In other early work on $\mathcal{PT}$-symmetric quantum mechanics Prof. Bender discovered new quasi-exactly solvable potentials. In quantum field theory Prof. Bender used perturbative, nonperturbative, and numerical techniques to study $\mathcal{PT}$-symmetric quantum electrodynamics and supersymmetric theories. He solved the Schwinger-Dyson equations and calculated Green's functions in various $\mathcal{PT}$-symmetric quantum field theories.

With all this intense research activity the field of $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians grew rapidly. However, a consensus had developed in the research community that non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians suffered from a serious drawback; namely, that the norm of the Hilbert space is not positive definite. The concern was that while $\mathcal{PT}$-symmetric theories are fascinating from a mathematical standpoint, it would be difficult to formulate a physical probabilistic theory of $\mathcal{PT}$-symmetric quantum mechanics with measurable observables.

In 2002 Prof. Bender and his collaborators D. Brody and H. Jones achieved a breakthrough that overcomes these problems. He discovered that when the $\mathcal{PT}$ symmetry is not spontaneously broken, the theory possesses a hidden symmetry represented by a linear operator $\mathcal{C}$ that commutes with the Hamiltonian. The operator $\mathcal{C}$ represents a new *observable quantum number* related to particle-antiparticle symmetry. In terms of $\mathcal{C}$, Prof. Bender constructed a $\mathcal{CPT}$ inner product that is associated with a *positive-definite* norm. This new $\mathcal{C}$ symmetry solves and eliminates the physical problems that were believed to plague $\mathcal{PT}$-symmetric theories. This discovery was published as a Physical Review Letter. This paper showed that $\mathcal{PT}$-symmetric theories are physical unitary quantum theories. There is a Hilbert space with a positive metric, a probabilistic interpretation, observables, and so on. The demonstration that non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians are consistent with the requirements of quantum mechanics is a significant development in the fundamental theory of quantum mechanics. In a 2004 PRL, Prof. Bender extended these results to quantum field theory.

Space-time reflection symmetry ($\mathcal{PT}$ symmetry) is *not* in conflict with conventional Hermiticity. It is merely a generalization and extension of ordinary quantum mechanics. For example, in the case of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon$, as ϵ tends to 0, the $\mathcal{PCT}$ norm reduces to the Hermitian conjugate norm of ordinary quantum mechanics. However, $\mathcal{PT}$-symmetric quantum mechanics differs from ordinary Hermitian quantum mechanics in several respects: In Hermitian quantum theory the parity operator $\mathcal{P}$ and the charge operator $\mathcal{C}$ commute, but in a $\mathcal{PT}$-symmetric theory they do not commute. As a result, the theory appears to have an energy spectrum for which the masses of particles and corresponding antiparticles may be different. (This result may account for the observed baryon-antibaryon asymmetry of the universe.)

Bender's work raises another intriguing possibility; namely, that $\mathcal{PT}$-symmetric quantum field theory may be able to describe the properties of the elusive Higgs particle. The problem with the conventional Hermitian model Lagrangian $L = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^4$ used to describe the Higgs sector in terms of spontaneous symmetry breaking is that in four dimensions this theory is not asymptotically free and is trivial. However, the $\mathcal{PT}$-symmetric Lagrangian $L = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - g\phi^4$ *is* asymptotically free and is a *nontrivial* theory in four-dimensional space-time. Furthermore, it is not necessary to introduce the notion of spontaneous symmetry breaking for this model because the one-point Green's function (the vacuum expectation value of the Higgs field ϕ) in the $\mathcal{PT}$-symmetric $-g\phi^4$ theory is *nonzero*.

Bender has shown that experimental predictions of $\mathcal{PT}$-symmetric Hamiltonians might be observed in condensed matter systems. Hermitian periodic potentials, such as $V(x) = \sin x$, exhibit energy bands and gaps. At one edge of each band of allowed energies the quantum-mechanical wave function is *bosonic* in character (it is 2π periodic), while at the other edge the wave function is *fermionic* in character (it is 4π periodic). In a Physics Letter [C. M. Bender, G. V. Dunne, and P. N. Meisinger, Phys. Lett. A **252**, 272 (1999)] Prof. Bender showed that a $\mathcal{PT}$-symmetric periodic potential, such as $V(x) = i\sin x$, also has real bands and gaps, but that there are half as many gaps. Moreover, at the edges of each band the wave function is always bosonic and never fermionic. It would be a difficult experiment to perform, but a laboratory observation of a material having this kind of band structure would be clear evidence of a system described by a $\mathcal{PT}$-symmetric Hamiltonian. Indeed some recent experimental work done in Vienna by A. Zeilinger on this Hamiltonian may lead to a clear experimental signature of non-Hermitian quantum systems.

A more dramatic demonstration of the observable difference between Hermitian and non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians was reported in a Physical Review Letter in 2007: "Faster than Hermitian Quantum Mechanics, C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. **98**, 040403 (2007). In this paper it was shown that non-Hermitian Hamiltonians give a faster-than-Hermitian time evolution of states in the Hilbert space. Thus, a quantum computer based on a non-Hermitian Hamiltonian can outperform a quantum computer based on a Hermitian Hamiltonian.

Bender's work on non-Hermitian Hamiltonians has already led to new directions in theoretical research. The work of Zamolodchikov, Dorey, Dunning, Tateo, Shin, Mostafazadeh, and others reveals that there is a deep connection between $\mathcal{PT}$ symmetry and integrable models, which is known as the *ODE/IM correspondence*. (See the review paper by Dorey et al, "The ODE/IM Correspondence," P. Dorey, C. Dunning, and R. Tateo, arXiv: hep-th/0703066.) It has been shown that methods normally used in conformal theories (Bethe *ansatz*, Baxter TQ relation, monodromy group methods, and so on) have direct applicability to these new kinds of non-Hermitian Hamiltonians. This is a fascinating new area of mathematical physics.

Bender's research has clarified some mysterious results regarding non-Hermitian Hamiltonians that were used in the past to describe interesting and nontrivial physical systems. For example, Wu showed that the ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [T. T. Wu, Phys. Rev. **115**, 1390 (1959)]. Wu found that the ground-state energy of this system is real and conjectured that all the energy levels were real. Hollowood showed that even though the Hamiltonian for a complex Toda lattice is non-Hermitian, the energy levels are real [T. Hollowood, Nucl. Phys. B **384**, 523 (1992)]. Non-Hermitian Hamiltonians of the form $H = p^2 + ix^3$ and cubic quantum field theories arise in studies of the Lee-Yang edge singularity [M. E. Fisher, Phys. Rev. Lett. **40**, 1610 1978; J. L. Cardy, *ibid.* **54**, 1345 1985; J. L. Cardy and G. Mussardo, Phys. Lett. B **225**, 275 1989; A. B. Zamolodchikov, Nucl. Phys. B **348**, 619 (1991)] and in various Reggeon field theory models [R. Brower, M. Furman, and M. Moshe, Phys. Lett. B **76**, 213 (1978); B. Harms, S. Jones, and C.-I Tan, Nucl. Phys. **171**, 392 (1980) and Phys. Lett. B **91**, 291 (1980)]. In each of these cases a non-Hermitian Hamiltonian having a real spectrum was confusing at the time, but now the explanation is simple. In each case the non-Hermitian Hamiltonian is $\mathcal{PT}$-symmetric.

A recent advance in theoretical physics that has resulted from Prof. Bender's work on $\mathcal{PT}$-symmetric quantum theory is the recognition that when quantum field theories possess *ghosts* (states of negative norm), which may arise when renormalizing the theory, it is because the Hamiltonian has become non-Hermitian. It is often the case that this Hamiltonian is $\mathcal{PT}$ symmetric, and if it is quantized using the methods of $\mathcal{PT}$-symmetric quantum mechanics, the so-called ghost states become conventional physical states having positive norm.

A case in point is the Lee model, which was proposed by T. D. Lee in 1954 as a quantum field theory in which the entire renormalization program could be carried out exactly and in closed form. One year later, Pauli and Källén argued in a famous paper that upon renormalization, the model develops a new negative-norm state that makes the S matrix nonunitary. The problem of how to treat this ghost state was studied by many well known physicists, but it remained unsolved for 50 years and the prevailing belief was that the Lee model was fundamentally flawed. However, in 2005 Prof. Bender solved this 50-year-old problem:

"Ghost Busting: $\mathcal{PT}$-Symmetric Interpretation of the Lee Model,"
C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang
Physical Review D **71**, 025014 (2005), arXiv hep-th/0411064

This paper shows that the renormalized Lee-model Hamiltonian is $\mathcal{PT}$-symmetric. Bender calculated the $\mathcal{C}$ operator exactly and demonstrated that the so-called ghost state is actually a physically acceptable state and that the S matrix is, in fact, unitary. Recently, other physicists, such as Smilga et al and Curtright et al, have followed Prof. Bender's approach and have shown that some field theories that were thought to contain states of negative norm ("ghosts") actually have only physically acceptable positive-norm states when the theory is properly quantized.

In a recently submitted paper Bender and Mannheim resolved yet another 50-year-old problem involving the "ghost" states in the Pais-Uhlenbeck model:

"No-ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model"
C. M. Bender and P. D. Mannheim, arXiv: hep-th/0706.0207

Here again, the Hamiltonian for this higher-derivative model is not Hermitian but it *is* $\mathcal{PT}$-symmetric. The long-standing belief that this model possesses states of negative norm is false. This result suggests that there may not be a problem with quantizing quantum field theories having higher-derivative field equations.

In the coming year Prof. Bender will continue his research in the area of $\mathcal{P}$-symmetric quantum theory. Much more work needs to be done on the problem of investigating the appearance of ghosts in conventional quantum theories to determine whether some of the quantum theories that have been discarded for being plagued by ghosts are in fact valid quantum theories that possess $\mathcal{PT}$ symmetry. Additional work needs to be done on the $\mathcal{PT}$-symmetric version of conventional quantum theories. For example, additional research must be done on $\mathcal{PT}$-symmetric quantum electrodynamics:

"A Nonunitary Version of Massless Quantum Electrodynamics Possessing a Critical Point"
C. M. Bender and K. A. Milton
J. Phys. A: Math. Gen. **32**, L87-L92 (1999)

"$\mathcal{PT}$-Symmetric Quantum Electrodynamics"
C. M. Bender, I. Cavero-Pelaez, K. A. Milton, and K. V. Shajesh
Physics Letters B **613**, 97 (2005) [arXiv: hep-th/0501180]

Also, Prof. Bender will investigate models of $\mathcal{PT}$-symmetric gravity. Among the many investigations to be conducted include studies of anomalies in $\mathcal{PT}$-symmetric quantum field theory and understanding the nature of complex $\mathcal{PT}$-symmetric functional integrals. Benders early investigations include:

"Equivalence of a Complex $\mathcal{PT}$-Symmetric Quartic Hamiltonian and a Hermitian Quartic Hamiltonian with an Anomaly"
C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie
Phys. Rev. D **74**, 025016 (2006) [arXiv: math-ph/0605066].

Already, some work has been done on the topics of lattice evaluation of $\mathcal{PT}$-symmetric functional integrals (C. Bernard and V. Savage, H. F. Jones and R. J. Rivers), large-N approximations (M. C. Ogilvie and P. N. Meisinger), $\mathcal{PT}$-symmetric matrix models (M. C. Ogilvie and P. N. Meisinger, J. Feinberg).

Finally, it should be noted that $\mathcal{PT}$-symmetric classical mechanics and classical field theory is a rich and interesting open subject that needs to be investigated in great detail. Prof. Bender has shown that nearly all of the well-known and heavily-studied nonlinear wave equations (Korteweg-de Vries, generalized Korteweg-de Vries, Camassa-Holm, Sine-Gordon, Boussinesq, for example) are $\mathcal{PT}$-symmetric. Prof. Bender's early investigations of classical $\mathcal{PT}$-symmetric theories include:

"Complex Trajectories of a Simple Pendulum"
C. M. Bender, D. D. Holm, and D. W. Hook
J. Phys. A: Math. Theor. **40**, F81-F89 (2007) [arXiv: math-ph0609068]

"$\mathcal{PT}$-Symmetric Extension of the Korteweg-de Vries Equation"
C. M. Bender, D. C. Brody, J.-H. Chen, and E. Furlan
J. Phys. A.: Math. Theor. **40**, F153 (2007) [arXiv: math-ph0610003]

"Spontaneous Breaking of Classical $\mathcal{PT}$-Symmetry"
C. M. Bender and D. W. Darg
J. Math. Phys. **48**, 042703 (2007) [arXiv:hep-th/0703072]

"Complexified Dynamical Systems"
C. M. Bender, D. D. Holm, and D. W. Hook
J. Phys. A: Math. Theor. **40**, F793-F804 (2007) [arXiv: hep-th/0705.3893]

"Does the complex deformation of the Riemann equation exhibit shocks?"
C. M. Bender and J. Feinberg
Submitted

Much more work needs to be done. For example, the question of the complex extension of the transition to chaos is an interesting and open problem and there is currently work in progress with J. Feinberg and D. W. Hook on the complex chaotic transition of the kicked rotor.

7. Miscellaneous work in mathematical physics

Because of his strong background in applied mathematics, Prof. Bender is able to enter quickly into fruitful research collaborations with a wide variety of scientists whose interests range from applied to theoretical. This past year Prof. Bender has worked on the optimal design of large-scale computing arrays

"Optimal Shape of a Blob"
C. M. Bender and M. A. Bender
J. Math. Phys. **48**, 073518 (2007) [arXiv: math-ph/0703025]

He also has worked on the formulation of general orthogonal polynomials in terms of nonlinear integral equations

"Nonlinear Integral-Equation Formulation of Orthogonal Polynomials"

C. M. Bender and E. Ben-Naim

J. Phys. A: Math. Theor. **40**, F9-F15 (2007) [arXiv: math-ph/0610071]

Work is currently in progress with R. Teodorescu and E. Ben-Naim on such asymptotics problems as the double-scaling limit of orthogonal polynomials.

Concluding remarks

Bender's work is characterized by great originality and contributions that do not follow conventional trends. Measured by the number of people who have used the mathematical techniques that Prof. Bender has developed and have worked in areas of research that he has initiated, such as coupling-constant analyticity, large-order behavior of perturbation theory, and $\mathcal{PT}$-symmetric quantum mechanics, many of Prof. Bender's papers have been seminal and have had and continue to have great impact in mathematical physics. Prof. Bender has over 250 published papers including 21 Physical Review Letters and his published work has over 10,000 citations.

Prof. Bender has already published eight papers in 2007 including a Physical Review Letter. In 2007 he was a principal invited speaker at eight international conferences and he gave twelve invited talks at universities and laboratories.

Prof. Bender has a long list of recognitions. He is a Fellow of the American Physical Society and the UK Institute of Physics. He has received Guggenheim, Fulbright, Lady Davis, Sloan, and Rockefeller Foundation Bellagio Fellowships. He was the Ulam Scholar at Los Alamos National Laboratory for the 2006-07 academic year. He is currently serving a five-year term as Editor-in-Chief of the Journal of Physics A and in the past he has served on the editorial boards for several other journals.