Spin and measurement.

Spin $\frac{1}{2}$: eigenstates of $S_z$ and $S^2$

Eigenvalues: $\pm \frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$

$S_z$ is an observable. Something we can measure with Stern-Gerlach apparatus.

But we can also arrange to arrange to measure $S_x$ or $S_y$ - what do we learn?

1) Note that any possible state can be written as a superposition of $S_z$ eigenstates:

$|\uparrow\rangle = \chi_+ = (\frac{1}{\sqrt{2}})$, $\chi_- = |\downarrow\rangle = (\frac{1}{\sqrt{2}})$

$|\psi\rangle = \chi = (a, b)$ where $a, b$ are complex numbers.

If we measure $S_z$, the probability of outcome $+\frac{1}{2}$ or $-\frac{1}{2}$ are easy to determine. (try the $|a|^2$ or $|b|^2$)

2) If we measure $S_x$, then we need to first express $|\psi\rangle$ as a superposition of eigenstates of $S_x$.

Thus we have $\chi_+^{(x)} = (\frac{\sqrt{2}}{2})$, $\chi_-^{(x)} = (\frac{\sqrt{2}}{2})$
example: suppose a particle is in state

$$|\psi\rangle = \Psi = \frac{1}{\sqrt{6}} \left( \frac{1+i}{2} \right)$$

- check that this is normalized

$$\langle \psi | \psi \rangle = \Psi^\dagger \Psi = \frac{1}{\sqrt{6}} \left( \frac{1+i}{2} \right)^\dagger \frac{1}{\sqrt{6}} \left( \frac{1+i}{2} \right)$$

$$= \frac{1}{6} (2+4) = 1$$

- if you measure Sz what is the probability of getting $\frac{+\hbar}{2}$?

$$\left| \frac{1+\hbar}{\sqrt{6}} \right|^2 = \frac{1}{3}$$

(probability of getting $-\frac{\hbar}{2}$ is $\left| \frac{-\hbar}{\sqrt{6}} \right|^2 = \frac{2}{3}$)

- what is the probability of measuring $\frac{+\hbar}{2}$ for $S_x$?

→ need to express $\Psi$ in $S_x$ basis.

$$|\psi\rangle = a \chi_+ + b \chi_-$$

$$\frac{1}{\sqrt{6}} \left( \frac{1+i}{2} \right) = a \left( \frac{1}{\sqrt{2}} \right) + b \left( \frac{-1+i}{\sqrt{2}} \right)$$

→ solve for $a$ and $b$.

$$a = \chi_+^\dagger \chi$$

$$= \left( \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{6}} \frac{1+i}{2} \right) = \frac{1}{\sqrt{12}} (1+i+2)$$

$$= \frac{3+i}{\sqrt{12}}$$

$$b = \chi_-^\dagger \chi$$

$$= \left( \frac{1}{\sqrt{2}} \frac{-1+i}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{6}} \frac{-1+i}{2} \right) = \frac{-1+i}{\sqrt{12}}$$
The probability of getting $+\frac{5}{2}$ along $S_x$ is

$$|a|^2 = \frac{10}{12} = \frac{5}{6} \quad \text{(prob of } -\frac{5}{2} \text{ is } \frac{1}{6})$$

What is $\langle S_x \rangle$?

$$\frac{5}{6} \cdot \frac{5}{2} + \frac{1}{6} \cdot \frac{-5}{2} = \frac{4}{12} = \frac{1}{3}$$

We can do this more directly...

$$\langle S_x \rangle = \chi^+ S_x \chi = \left(\begin{array}{c} \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{array}\right) \begin{pmatrix} 0 & +\frac{5}{2} \\ \frac{5}{2} & 0 \end{pmatrix} \left(\begin{array}{c} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{array}\right) \begin{pmatrix} \frac{2.5}{2.6} \\ \frac{1+0.5}{2.6} \end{pmatrix}$$

$$= \frac{1}{3}$$

$\rightarrow$ video: quantum computer animated.

**Measurement story:**

**K:** I have a particle in state $|+\rangle$

**S:** What's the $z$ component of its spin?

**K:** $+\frac{5}{2}$

**S:** Okay, what's the $x$-component?

**K:** All I can say is that if you measure $S_x$ there is a 50/50 chance of getting $+\frac{5}{2}$

**S:** You mean you don't know?

**K:** I do know exactly, the state $|+\rangle$ it doesn't have a particular $S_x$.
(Student takes the spin to portale and measures $S_z$)

S: I get the result $+\frac{1}{2}$... ha! See it does have a well defined x-component.

K: Yes, but you messed it up... if we measure $S_z$ again we'd get $+\frac{1}{2}$ with 50% chance...

Bloch Sphere:

\[
\ket{\uparrow} + \ket{\downarrow} \equiv \frac{1}{\sqrt{2}}
\]

\[
\langle S_z \rangle = +\frac{1}{2} \quad \Delta S_z = 0 \quad \langle S_z \rangle = 0
\]

\[
\langle S_x \rangle = 0 \quad \Delta S_x = \pm \frac{1}{2} \quad \langle S_x \rangle = +\frac{1}{2}
\]

Start with $\ket{\uparrow}$ and measure $S_z$: get $+\frac{1}{2}$.
IDENTICAL PARTICLES

Electrons are identical particles: even God can't tell them apart.

How do we describe the wave function of identical particles?

Two particles:

\[ \psi(r_1, r_2) = \psi(x_1, y_1, z_1, x_2, y_2, z_2) \]

Schrödinger equation:

\[ \left( \frac{-\hbar^2}{2m} \nabla^2 + \frac{e^2}{2r_{12}} \right) \psi + U(r_1, r_2) = E \psi \]

Our approach to solving this would be to separate variables, but in general \( U(r_1, r_2) \) is not separable.

Ex: Coulomb potential \( U(r_1, r_2) = \frac{(e)^2}{4\pi\varepsilon_0} \frac{1}{|r_1 - r_2|} \)

Let's simplify: consider two non-interacting identical particles with \( U=0 \) ... then we can separate variables.

**Example:** 2 particles in a 1D box

Let's say that one is in state \( n=4 \)

and the other is in state \( n=3 \)

The wave function is then a product of the two single particle wavefunctions

\[ \psi(x_1, x_2) = \Psi_3(x_1) \Psi_4(x_2) = \frac{2}{L} \sin \left( \frac{3\pi x_1}{L} \right) \sin \left( \frac{4\pi x_2}{L} \right) \]

The probability of finding a particle at a certain location

\[ P(x_1, x_2) = |\psi(x_1, x_2)|^2 = \frac{4}{L^2} \sin^2 \left( \frac{3\pi x_1}{L} \right) \sin^2 \left( \frac{4\pi x_2}{L} \right) \]

At location \( \frac{L}{2} \), the probability of finding a particle 1
is zero, but finite for particle 2, thus we could distinguish between the two particles and they are not indistinguishable.

In order to make them indistinguishable we take a combination of both particles.

\[
\psi(x_1, x_2) = \frac{\sqrt{2}}{L} \sin\left(\frac{4\pi x_1}{L}\right) \sin\left(\frac{3\pi x_2}{L}\right) + \frac{\sqrt{2}}{L} \sin\left(\frac{4\pi x_2}{L}\right) \sin\left(\frac{3\pi x_1}{L}\right)
\]

\[
= \psi_4(x_1) \psi_3(x_2) + \psi_4(x_2) \psi_3(x_1)
\]

\[
P(x_1, x_2) = \psi_4^2(x_1) \psi_3^2(x_2) + \psi_4^2(x_2) \psi_3^2(x_1) - 2 \psi_4(x_1) \psi_3(x_2) \psi_4(x_2) \psi_3(x_1)
\]

This is symmetric under interchange of \(x_1\) and \(x_2\) so the particles are now indistinguishable.

Note: both the symmetric combination \(\psi_4 \psi_3 + \psi_4 \psi_3\)

and the antisymmetric combination \(\psi_4 \psi_3 - \psi_4 \psi_3\)

are not in indistinguishable particles.

This has profound consequences. (like imaginary #s)

Example:

Two identical particles in a box (1-D infinite square well) with total energy \(\frac{5\pi^2 \hbar^2}{2mL^2}\) and prob density, determine the two particle wavefunction and explore the consequences of a symmetric vs. anti-symmetric linear combination.

\[
E = (n^2 + m^2) \frac{\pi^2 \hbar^2}{2mL^2} \Rightarrow n=1 \ m=2
\]
\[ y(x_1, x_2) = A \sin \left( \frac{\pi x_1}{L} \right) \sin \left( \frac{2\pi x_2}{L} \right) \pm A \sin \left( \frac{\pi x_2}{L} \right) \sin \left( \frac{2\pi x_1}{L} \right) \]

Probability density:
\[
P(x_1, x_2), |y|^2 = |A|^2 \left( \sin^2 \left( \frac{\pi x_1}{L} \right) \sin^2 \left( \frac{2\pi x_2}{L} \right) + \sin^2 \left( \frac{\pi x_2}{L} \right) \sin^2 \left( \frac{2\pi x_1}{L} \right) \\
+ 2 \sin \frac{\pi x_1}{L} \sin \frac{2\pi x_2}{L} \sin \frac{\pi x_2}{L} \sin \frac{2\pi x_1}{L} \right)
\]

\(\Rightarrow \text{plot this probability in mathematica.}\)

\[ x_2 \]
\[ x_1 \]

\[ \text{symmetric} \quad \text{anti-symmetric} \]

Look at the probability waves \( x_1 = x_2 = x \)

\[
P_s = \psi^* \psi = 4 A^2 \sin^2 \left( \frac{\pi x}{L} \right) \sin^2 \left( \frac{2\pi x}{L} \right)
\]

\[ P_s \]

\[ P_A = \psi_A^* \psi_A = A^2 2 \sin^2 \left( \frac{\pi x}{L} \right) \sin^2 \left( \frac{2\pi x}{L} \right) - 2 \sin^2 \left( \frac{\pi x}{L} \right) \sin^2 \left( \frac{2\pi x}{L} \right) \\
= 0 \quad !
\]

Absolutely no probability of finding the two particles at the same location.

The symmetry of the linear combination drastically changes where the particles can be.
Let me emphasize that the particles are now integers.

**Spin-statistics theorem**

- **Half-integer spin** particles are fermions and form anti-symmetric linear combination.
- **Integer spin** particles are bosons and form symmetric combination.

This will affect the probability of the multi-particle state.

**Pauli exclusion principle**

Two fermions cannot occupy the same spin-particle state.

Why? \( \psi_n = \psi(x_1)\psi(x_2) - \psi(x_2)\psi(x_1) = 0 \)

Composite particles: eg. hydrogen atom, eg. protons, neutrons (fermions are composed of quarks)

These can be bosons or fermions.

We need to figure out the total spin of a composite particle. (Next week)

Proton \( \uparrow \) spin \( \frac{1}{2} \)

Neutron \( \downarrow \) spin \( \frac{1}{2} \)

Electron \( \uparrow \) spin 1
\[ \langle x^2 \rangle = \int x^2 \delta(x) \, dx = \int x^2 \delta(x-a) \, dx = a^2 \]

Case 1: distinguishable particles

Calculate the expectation value of the square of the distance between two particles.

Now, we'll explore the consequences of this distribution.

Exchange forces: we saw that we can use a symmetric anti-symmetric linear combination for two particles.
We define
\[ |x\rangle \equiv \frac{1}{\sqrt{Z}} \int \varphi(x) \, dx \]

\[ \int_0^a \langle x | \varphi(x) \rangle \, dx = \int_0^a \frac{\varphi(x)}{\sqrt{Z}} \, dx \int_0^a \frac{\varphi(x')}{\sqrt{Z}} \, dx' \]

\[ \frac{1}{Z} \int_0^a \langle x | \varphi(x) \rangle \, dx \int_0^a \frac{\varphi(x')}{\sqrt{Z}} \, dx' = \frac{1}{Z} \int_0^a \langle x | \varphi(x) \rangle \, dx \int_0^a \frac{\varphi(x')}{\sqrt{Z}} \, dx' \]

Not necessarily into this form.

Next, we need to calculate
\[ \left[ \langle 2x | + \langle 2x' | \right] \frac{Z}{2} = \langle 2x | \frac{Z}{2} = \langle 2x | \frac{Z}{2} \]

by the same argument.

Now: Case II: Identical particles: do both symmetric and antisymmetric at the same time.
For a geometric construction of the symmetrical requirement, we have a geometric force, which is not merely a force of exchange. Force will be non-exchange.

The free exchange term

The difference comes from the integral

\[ \int x^a (x^b (x^c x^d) x^e) dx. \]

The difference comes from this integral

\[ \int x^a x^b x^c x^d x^e dx. \]

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