Physics 318 Introduction to Quantum Physics II  
Instructor: Kater Murch  
Due Wednesday March 9, in class

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Problem Set 6 (50 points)

Suggested Reading: Class notes, Harris: 9.1-9.5, Schroeder: 6.1, 6.2

1. **White dwarf star** (25 points) A white dwarf star is essentially a degenerate electron gas, with a bunch of nuclei mixed in to balance the charge and to provide the gravitational attraction that holds the star together. In this problem, you will derive a relationship between the mass and the radius of a white dwarf star modeling the star as a uniform density sphere. White dwarf stars tend to be extremely hot, but it is nevertheless an excellent approximation to set $T = 0$.

(a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass $M$, radius $R$) must equal:

$$
U_{\text{grav}} = -(\text{constant}) \frac{GM^2}{R}.
$$

(1)

You can also explicitly calculate the negative work needed to assemble the sphere shell by shell to show that the constant is $3/5$, but this is not necessary.

(b) Assuming that the star contains one proton and one neutron for each electron, and that the electrons are nonrelativistic, show that the total (kinetic) energy of the degenerate electrons equals,

$$
U_{\text{KE}} = (\text{const.}) \frac{4\pi^2 \hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}.
$$

(2)

Where (const.) is a constant that can be expressed exactly in terms of $\pi$ and cube roots and such. Determine this constant.

(c) The equilibrium radius of the white dwarf is that which minimizes the total energy $U_{\text{grav}} + U_{\text{KE}}$. Sketch (or plot) the total energy as a function of $R$ and determine the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?

(d) Evaluate the equilibrium radius for $M = 2 \times 10^{30}$ kg, the mass of the sun. Evaluate the density, how does it compare to the density of water?

(e) Calculate the Fermi energy and Fermi temperature ($E_F/k = T_F$). Is the approximation $T = 0$ valid?

(f) Suppose that the electrons are instead highly relativistic. Using the result of problem 3 ($E_F \propto (N/V)^{1/3}$), show that the total kinetic energy of the electrons is now proportional to $1/R$ rather than $1/R^2$ and there is no stable equilibrium radius for such a star.

(g) The transition from the non-relativistic to the ultra-relativistic regime occurs where the average kinetic energy of an electron is equal to its rest energy $mc^2$. Is the non-relativistic approximation valid for a one-solar-mass white dwarf? Above what mass would the white dwarf become relativistic and hence unstable?

2. **Magnons** (25 points) A ferromagnet is a material like iron that magnetizes spontaneously even in the absence of an externally applied magnetic field. At $T = 0$ the magnetization has the maximum possible value, with all dipoles lined up. If all the dipoles are lined up, then the
total magnetization is $2\mu_B N$, where $\mu_B$ is the Bohr magneton. At higher temperatures, the excitations are spin waves. Like sound waves, spin waves are quantized, each wave mode can have integer multiples of a basic energy unit. Each energy unit is a magnon. Each magnon reduces the magnetization by $\sim \mu_B$. However, whereas the frequency of a sound wave is inversely proportional to its wavelength, the frequency of a spin wave is proportional to $1/\lambda^2$.

In analogy to the energy-momentum of a non-relativistic particle, we can write the energy of a magnon as $E = \hbar^2 k^2 / 2m^*$, where $k = 2\pi / \lambda$ and $m^*$ is a constant related to the spin-spin interaction energy and the atomic spacing and can be thought of as an effective mass for the magnon.

a) Show that at low temperatures, the number of magnons per unit volume in a three dimensional ferromagnet is given by,

$$N_m/V = 2\pi \left( \frac{2m^* kT}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx$$

Evaluate the integral numerically.

b) Use this result to find an expression for the fractional reduction in magnetization $(M(0) - M(T))/M(0)$. Write your answer in the form $(T/T_0)^{3/2}$ and estimate $T_0$ for iron where $m^* = 1.24 \times 10^{-29}$.

c) Calculate the heat capacity due to magnetic excitations in a ferromagnet at low temperature. You should find $C_V/Nk = (T/T_1)^{3/2}$ where $T_1$ differs from $T_0$ by only a numerical constant. Estimate $T_1$ for iron and compare the magnon and phonon contributions to the heat capacity. (The Debye temperature of iron is 470 K).